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LETTER TO THE EDITOR

On the structure of phase-space, Hamiltonian variables and statistical approach to the description of two-dimensional hydrodynamics and magnetohydrodynamics

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Abstract. The peculiarities of the Hamiltonian description of 2D hydrodynamics and magnetohydrodynamics are discussed in connection with recent attempts to construct a thermodynamical picture for 2D turbulence. The unconstrained Hamiltonian variables are displayed in both cases and the role of topology of the flow in tentative statistical equilibria is discussed. As a by-product the Clebsch-type variables for 2D magnetohydrodynamics are obtained describing flows with initial zero vorticity.

The recent papers of Miller [1] and Robert and Sommeria [2] revived an interest in the thermodynamical description of 2D turbulence. This approach has a long history starting from the classical work of Onsager [3] on statistics of point vortices (for a review of the activity up to 1980 see [4]). Putting aside a proof of ergodicity (if any), the dominant problem arising in statistics of continuous vorticity distributions within the Eulerian framework is how to properly take into account an infinity of integrals of motion in a partition function. On the intuitive level one expects something like an infinite product of Dirac delta-functions of conservation laws to be introduced into the Gibbs measure but it is hard to attribute an operational meaning to this idea. The papers [1, 2] contain interesting attempts to do that based on certain *a priori* statistical hypotheses.

The standard way of introducing a statistical description for a dynamical system is to use a Hamiltonian formalism which allows one to build a statistical measure with the help of a Hamiltonian and a volume element of phase-space. So in this context it is worth trying to understand what are the peculiarities of a Hamiltonian description for a 2D fluid. It turns out that it is very far from the familiar picture of a flat phase-space formed by canonical coordinates and momenta and it is the purpose of this letter to draw attention to this fact. The structure of a phase-space will be displayed and true (unconstrained) Hamiltonian variables will be demonstrated for both 2D hydrodynamics (HD) and 2D magnetohydrodynamics (MHD) since historically the thermodynamical approach was exercised in parallel for these two systems (see [4]).

We start with 2D HD written in the form of the vorticity equation

$$\dot{\omega}+J(\omega,\psi)=0 \tag{1}$$

where ω and ψ are vorticity and stream-function respectively, $\omega=\Delta\psi$ and J denotes the Jacobian. It is well known that an arbitrary function of vorticity integrated over the domain $\mathscr{D}$ of the flow is conserved

$$I_F = \int_{\mathscr{D}} F(\omega)\,\mathrm{d}x\,\mathrm{d}y = \text{constant} \tag{2}$$

which gives an infinite number of integrals of motion. The question of a Hamiltonian description of the dynamical system (1) is obviously non-trivial since we have a nonlinear equation of motion and a quadratic Hamiltonian kinetic energy of the fluid:

$$H = -\tfrac{1}{2}\int_{\mathscr{D}} \omega\psi\,\mathrm{d}x\,\mathrm{d}y. \tag{3}$$

The answer is given by a Poisson bracket

$$\{A[\omega], B[\omega]\} = -\int_{\mathscr{D}} \omega J\left(\frac{\delta A}{\delta\omega}, \frac{\delta B}{\delta\omega}\right)\mathrm{d}x\,\mathrm{d}y \tag{4}$$

which allows us to rewrite (1) in the standard Hamiltonian form

$$\dot{\omega} = \{\omega, H\}.$$

This Poisson bracket may be deduced from the underlying algebraic structure first pointed out by Arnold [15] and relating (1) to the Lie algebra of symplectic diffeomorphisms of $\mathscr{D}$, sdiff $\mathscr{D}$. This latter may be realized as an algebra of infinitesimal transformations of the form:

$$\delta\omega = J(\chi, \omega) \tag{5}$$

where χ is an arbitrary smooth function in $\mathscr{D}$. This evidently corresponds to an infinitesimal area-preserving coordinate transformation

$$(x, y) \to (x - \chi_y, y + \chi_x). \tag{6}$$

Hence, the global version of this equation may be immediately written:

$$(x, y) \to (X(x, y), Y(x, y)) \qquad J(X, Y) = 1 \tag{7}$$

giving in turn the global version of (5)

$$\omega(x, y) \to \omega(X(x, y), Y(x, y)) \qquad J(X, Y) = 1. \tag{8}$$

Equations (5) and (8) actually provide a co-adjoint representation of symplectic diffeomorphisms and (4) is a Kirillov bracket related to a Kirillov form (a condensed account of its theory may be found in [5], see also [6]). By construction this bracket is degenerate since any functional built of ω's and invariant under the group action (5) (a Casimir functional) annihilates the bracket identically. The integrals (2) are just of this geometric nature.

The bracket (4) becomes non-degenerate and, hence, defines the true Hamiltonian dynamics only on a certain submanifold of the original linear space of ω's. This manifold is a so-called co-adjoint orbit (see [7] for a discussion of co-adjoint orbits in a hydrodynamical context) and may be obtained starting from a sample vorticity pattern $\omega_0(x, y)$ (which may be thought of, but not necessarily, as initial data for a Cauchy problem for (1)) and implementing all possible transformations (8). It is evident that integrals (2) remain constant under such changes of variable. In this way the space of original vorticity variables is foliated [7] into symplectic manifolds (phase-spaces)—co-adjoint orbits defined by their representatives ω_0. These latter define in turn the values of Casimirs (2) fixed for the given orbit. All the points (vorticity fields) on such a manifold may be transformed into one another by smooth area-preserving changes of variables, i.e. a point on each of these phase-spaces is given by

$$\omega(x, y) = \omega_0(X(x, y), Y(x, y)) \qquad J(X, Y) = 1. \tag{9}$$

Vice versa, the points that cannot be connected by this symmetry transformation, e.g. those possessing different topological characteristics such as the structure of critical points, belong to the different phase-spaces. Parameters of the symmetry transformations, modulo those leaving ω_0 invariant, provide natural coordinates for such a phase-space and are the true non-constrained Hamiltonian variables for 2D ideal HD.

Unfortunately, there is no explicit parametrization for an arbitrary symplectic diffeomorphism. However for those not so far from identity the generating function method may be used. Namely, the function $S(x, Y)$ of old x- and new Y-coordinates may be intrdouced, such as

$$X = x + \frac{\partial S}{\partial Y} \qquad Y = y - \frac{\partial S}{\partial x}. \tag{10}$$

If the solvability conditions for Y as a function of x, y are satisfied, which is the case for the diffeomorphisms not very far from identity (a discussion of this point see e.g. in appendix 9 of [5]) then (10) does define a diffeomorphism and a function $S(x, y)$ provides its parametrization. Hence, those S which are functionally independent on ω_0 are the true Hamiltonian variables in this case.

Although the Hamiltonian variables just described seem rather far from being useful in practical calculations we can nevertheless draw with their help some conclusions about hypothetical thermodynamical equilibria. Indeed, any statistical measure contains a volume element of phase-space. This volume element is defined by symplectic structure which is, generally speaking, different for different orbits, i.e. flows with different topologies. Therefore, statistical measures are different for different orbits and on these grounds we do not expect universality in statistical equilibria (if of course they do exist) for flows with topologically different initial data. Moreover, if the standard argument [4] that a turbulent cascade is directed towards (inviscid) thermodynamical equilibrium is true we do not expect universality there as well.

Consider now 2D ideal MHD. The equations of motion written in terms of vorticity and magnetic potential $a(x, y)$ are:

$$\dot{\omega} + J(\omega, \psi) - J(\Delta a, a) = 0 \qquad \dot{a} + J(a, \psi) = 0. \tag{11}$$

The Hamiltonian is an energy functional:

$$H = -\tfrac{1}{2}\int_{\mathscr{D}} (\omega\psi + a\Delta a)\,\mathrm{d}x\,\mathrm{d}y \tag{12}$$

and a Poisson structure follows from the known (see [8]) relation between (11) and a semi-direct product of sdiff $\mathscr{D}$ and algebra of smooth functions $F(\mathscr{D})$. An infinitesimal co-adjoint action analogous to (5) may be deduced from this fact:

$$\delta\omega = J(\chi, \omega) + J(\sigma, a) \qquad \delta a = J(\chi, a) \tag{13}$$

where χ and σ are arbitrary smooth functions. This is enough to built a Kirillov bracket and, hence, a Poisson structure:

$$\{A[\omega, a], B[\omega, a]\}$$
$$= -\int_{\mathscr{D}} \left[\omega J\left(\frac{\delta A}{\delta\omega}, \frac{\delta B}{\delta\omega}\right) + a\left[J\left(\frac{\delta A}{\delta\omega}, \frac{\delta B}{\delta a}\right) + J\left(\frac{\delta A}{\delta a}. \frac{\delta B}{\delta\omega}\right)\right]\right]\mathrm{d}x\,\mathrm{d}y. \tag{14}$$

Again, an infinite number of Casimirs invariant under the transformation (13) and annihilating identically the Poisson bracket are present:

$$I_{F,G} = \int_{\mathscr{D}} [F(a) + \omega G(a)]\, \mathrm{d}x\, \mathrm{d}y = \text{constant} \tag{15}$$

where F and G are arbitrary functions of magnetic potential.

However the realization of the symmetry in terms of just coordinate transformations is now impossible and a global version of (13) is not obvious. To find it we shall use another realization of the underlying symmetry. The latter is most easily given in terms of an auxiliary complex function $f(x, y)$ which transforms as

$$f(x, y) \to \mathrm{e}^{\mathrm{i}\Lambda(x,y)} f(X(x, y), Y(x, y)) \qquad J(X, Y) = 1 \tag{16}$$

where the function $\Lambda(x, y)$ may be arbitrary. It is easy to check that generators of phase transformations and infinitesimal diffeomorphisms in this representation obey the same structure relations as generators of transformations (13). Equation (16) gives a fundamental representation of the symmetry group and the momentum map procedure (for a discussion of this subject within the present context see [6]) which allows us to get a co-adjoint representation from another (complex) representation and its conjugate may be used to get a following ansatz for ω and a:

$$\omega = \mathrm{i}J(f^*, f) a = f^* f. \tag{17}$$

It may be easily seen that under the infinitesimal version of (16) vorticity and magnetic potential transform according to (13) and we immediately get the global version of the latter from (16), (17):

$$\begin{aligned} &\omega \to \omega(X(x, y), Y(x, y)) - J(\Lambda(x, y), a(X(x, y), Y(x, y))) \\ &a \to a(X(x, y), Y(x, y)) \qquad J(X, Y) = 1. \end{aligned} \tag{18}$$

From now on we may forget about variables f^* and f. However it is worth mentioning that they provide Clebsch-like variables for 2D MHD and as usual (cf [6]) the Poisson bracket (14) may be recovered from the following canonical brackets for f and f^*:

$$\begin{aligned} &\{f^*(x, y), f(x', y')\} = \mathrm{i}\delta(x - x')\delta(y - y') \\ &\{f(x, y), f(x', y')\} = 0. \end{aligned} \tag{19}$$

The fact of positive-definiteness of a following from (17) poses no limitations since magnetic potential is defined up to an arbitrary constant. As often happens (see [6]) the ansatz (17) automatically selects a special kind of a co-adjoint orbit. Indeed the second term in the integrand in (15) vanishes identically in this case which means that 'a half' of possible integrals of motion are identically zero.

As to the structure of the phase-space, we again have a foliation of the original space of a and ω into co-adjoint orbits defined by their representatives a_0, ω_0. For any point on the orbit we have:

$$\begin{aligned} &\omega(x, y) = \omega_0(X(x, y), Y(x, y)) - J(\Lambda(x, y), a_0(X(x, y), Y(x, y))) \\ &a(x, y) = a_0(X(x, y), Y(x, y)) \qquad J(X, Y) = 1. \end{aligned} \tag{20}$$

It is clear from this equation that a singular orbit with $\omega_0 = 0$ corresponds to the situation described by Clebsch-like variables.

The genuine Hamiltonian variables are the function $\Lambda(x, y)$ and parameters of symplectic diffeomorphisms, modulo those leaving a_0 invariant. The conclusions about tentative thermodynamical equilibria hold on the same footing as in the case of pure HD.

So in the same way as canonical variables are coordinates of a phase-space in the standard Hamiltonian picture, the parameters of symplectic diffeomorphisms in 2D HD or those plus extra phase function in 2D MHD are the coordinates of a phase-space for these two systems. The crucial difference, however, is that now the phase-space is a curved (infinite-dimensional) manifold. It is geometry of such a manifold which plays the main role and which deserves further study.

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